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The global dynamics of a class of nonlinear vector fields in $\mathbb{R}^3$

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Abstract. In this paper, we study the geometric properties of a class of nonlinear polynomial vector fields in $\mathbb{R}^3$. By virtue of their induced vector fields, their global topological structures are discussed and we get that there are at least 82 types of invariant regions with different topological classification without considering the closed orbit. Finally, we give a sufficient condition of the existence of a closed orbit of the vector field.

Keywords: nonlinear polynomial system, tangent vector field, closed orbit.

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1 Introduction

It is very difficult to analyze the geometric properties of vector fields in $\mathbb{R}^3$ because their geometric properties are more complex than those of planar vector fields such as the strange attractor of the Lorenz equation. There are few results for vector fields in $\mathbb{R}^3$ such as the criterion of the existence of closed orbits, homoclinic and heteroclinic orbit, etc. However, most of the models of engineering and biology are higher dimensional systems [3]. Therefore, it is worth for us to investigate the vector fields in $\mathbb{R}^3$.

The simplest vector field in $\mathbb{R}^3$ is a linear homogeneous system, its local geometric properties were first analyzed by Reyn [12], and its global topological structure was given by Zhang and Liang [18]. For nonlinear vector fields in $\mathbb{R}^3$, Coleman in 1959 [5] first studied the geometric properties of flows of homogeneous vector fields in the neighborhood of the origin in $\mathbb{R}^3$. Later Sharipov [13] discussed the topological classifications of flows of homogeneous vector fields and gave seven types of different invariant cones of the homogeneous vector fields. Camacho in 1981 [1] investigated the topological classifications of the tangent vector fields induced by homogeneous vector fields of degree two in $\mathbb{R}^3$. Zhang et al. in 1999 [17] showed that there are at least sixteen types of different invariant cones by global topological analyses.

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More works about homogeneous systems can be seen in [2, 9, 10]. For non-homogeneous vector fields in $\mathbb{R}^3$, Llibre and Zhang [11] studied the polynomial first integrals for n-dimensional quasi-homogeneous system. Dumortier [6] used the quasi-homogeneous blow-up to investigate the singularity of planar systems. Zhang et al. [19] gave the global dynamics of a class of vector fields in $\mathbb{R}^3$. Huang and Zhao [8] studied the limit set of trajectories in a three-dimensional quasi-homogeneous system. More works about non-homogeneous systems can be seen in [14, 15, 16].

In this paper, we will investigate the following vector fields:

$$xF(x)+Q(x)\equiv(x_1F_1(x)+Q_1(x),x_2F_2(x)+Q_2(x),x_3F_3(x)+Q_3(x)), \tag{1.1}$$

where $x=(x_1,x_2,x_3)\in\mathbb{R}^3$, and

$$\begin{aligned}
Q_1(\lambda^{\alpha_1}x_1,\lambda^{\alpha_2}x_2,\lambda^{\alpha_3}x_3)&=\lambda^{\alpha_1-1+\delta}Q_1(x_1,x_2,x_3),\\
Q_2(\lambda^{\alpha_1}x_1,\lambda^{\alpha_2}x_2,\lambda^{\alpha_3}x_3)&=\lambda^{\alpha_2-1+\delta}Q_2(x_1,x_2,x_3),\\
Q_3(\lambda^{\alpha_1}x_1,\lambda^{\alpha_2}x_2,\lambda^{\alpha_3}x_3)&=\lambda^{\alpha_3-1+\delta}Q_3(x_1,x_2,x_3),\\
F_i(\lambda^{\alpha_1}x_1,\lambda^{\alpha_2}x_2,\lambda^{\alpha_3}x_3)&=\lambda^{m}F_i(x_1,x_2,x_3),\\
\frac{F_1(x)}{\alpha_1}=\frac{F_2(x)}{\alpha_2}&=\frac{F_2(x)}{\alpha_2}=f(x),\\
\lambda\in R,\quad &\delta,\alpha_1,\alpha_2,\alpha_3\in R^+.
\end{aligned} \tag{1.2}$$

In Section 2, we set up a bridge between the vector field $xF(x)+Q(x)$ in $\mathbb{R}^3$ and the tangent vector field $Q_T(u)$ on the two-dimensional manifold $S^2=\{u=(u_1,u_2,u_3):u_1^2+u_2^2+u_3^2=1\}$. In Section 3, we first discuss the relationship between the singular point of the vector field $xF(x)+Q(x)$ in $\mathbb{R}^3$ and the tangent vector field $Q_T(u)$ in S^2. In Section 4, we give the classification of the vector field $xF(x)+Q(x)$ and we prove that the vector field $xF(x)+Q(x)$ has at least 82 types of different topological classification without considering the number of limit cycles. At last we obtain the sufficient condition of the existence of closed orbit of the vector field $xF(x)+Q(x)$ in $\mathbb{R}^3$.

2 Global properties of $xF(x)+Q(x)$

In this section, we will investigate the global properties of $xF(x)+Q(x)$. For each $x\in\mathbb{R}^3\setminus\{(0,0,0)\}$, we make a transformation:

$$x=(x_1,x_2,x_3)=(r^{\alpha_1}u_1,r^{\alpha_2}u_2,r^{\alpha_3}u_3),\quad u=(u_1,u_2,u_3)\in S^2,\quad r\in\mathbb{R}^+$$

then vector field (1.1) in $\mathbb{R}^3\setminus\{(0,0,0)\}$ turns into

$$\begin{aligned}
\frac{dr}{dt}&=\frac{r^{m+1}\langle u,v\rangle+r^{\delta}\langle u,Q(u)\rangle}{\langle\overline{u},u\rangle},\\
\frac{du}{dt}&=\frac{r^{\delta-1}(\langle\overline{u},u\rangle Q(u)-\langle u,Q(u)\rangle\overline{u})}{\langle\overline{u},u\rangle}.
\end{aligned} \tag{2.1}$$

where $\overline{u}=(\alpha_1u_1,\alpha_2u_2,\alpha_3u_3)$, $v=(u_1F_1,u_2F_2,u_3F_3)$, $\langle\cdot,\cdot\rangle$ is the Euclidean inner product.

Introducing a new time τ by means of relation $d\tau = \frac{r^{\delta-1}}{\langle \overline{u}, u\rangle} dt$ (the time variable is still denoted by t), we could obtain

$$\frac{dr}{dt} = r^{m+2-\delta}\langle u, v\rangle + r\langle u, Q(u)\rangle \overset{D}{=\!=} r^{m+2-\delta}G(u) + rR(u), \tag{2.2a}$$

$$\frac{du}{dt} = \langle \overline{u}, u\rangle Q(u) - \langle y, Q(u)\rangle \overline{y} \overset{D}{=\!=} Q_T(u). \tag{2.2b}$$

where $G(u) = \langle u, v\rangle$, $R(u) = \langle u, Q(u)\rangle$. The vector field (2.2b) is called the tangent vector field of (1.1), and it is an independent system on S^2.

Proposition 2.1. *The flows of the vector field $xF(x) + Q(x)$ in $\mathbb{R}^3$ are topologically equivalent to the flows of system (2.2).*

3 Global geometric properties of $xF(x) + Q(x)$

In this paper, we only discuss the geometric properties for case $m + 1 - \delta > 0$ and the geometric properties for case $m + 1 - \delta < 0$ is similar to case $m + 1 - \delta > 0$. For convenience of the following discussion, we first introduce several notations. We will write g, γ, θ, Ω_γ, A_γ to denote a singular point, a trajectory, a closed orbit, an ω-, α-limit set of the trajectory γ of the vector field $Q_T(u)$ on sphere S^2, respectively. If we use the notation $S(l) = \{(\lambda^{\alpha_1}x_1, \lambda^{\alpha_2}x_2, \lambda^{\alpha_3}x_3) \mid (x_1, x_2, x_3) \in l \subset \mathbb{R}^3, \lambda \in \mathbb{R}^+\}$ (l may be a point or a curve), then $S(\gamma) = \{x \mid x \in \omega_\gamma, r_0 \in \mathbb{R}^+\}$. We will write w, Ω_w, A_w to denote a trajectory, an ω-, α-limit set of the trajectory w of $xF(x) + Q(x)$ on $S(\gamma)$, write θ^* to denote a closed orbit of $xF(x) + Q(x)$ on $S(\theta)$. At first we give some basic properties between the vector fields $xF(x) + Q(x)$ and $Q_T(u)$.

Theorem 3.1. *If $E(x_1, x_2, x_3)$ is a singular point of system (1.1), then $g = (x_1/r^{\alpha_1}, x_2/r^{\alpha_2}, x_3/r^{\alpha_3})$ is a singular point of (2.2b), where r satisfies $r^{-2\alpha_1}x_1^2 + r^{-2\alpha_2}x_2^2 + r^{-2\alpha_3}x_3^2 = 1$.*

Proof. If $E(x_1, x_2, x_3)$ is a singular point of system (1.1), then E satisfies $EF(E) + Q(E) = 0$, or

$$x_iF_i(x_1, x_2, x_3) + Q_i(x_1, x_2, x_3) = 0, \quad \text{i.e.} \quad \alpha_i x_i f(x_1, x_2, x_3) + Q_i(x_1, x_2, x_3) = 0,$$

then we have

$$\begin{aligned}
&\langle \overline{g}, g\rangle Q_i(g) - \langle g, Q(g)\rangle \overline{g_i} \\
&= \left(\frac{\alpha_1 x_1^2}{r^{2\alpha_1}} + \frac{\alpha_2 x_2^2}{r^{2\alpha_2}} + \frac{\alpha_3 x_3^2}{r^{2\alpha_3}}\right) r^{-\alpha_i+1-\delta} Q_i(x_1, x_2, x_3) - \left[\frac{x_1}{r^{\alpha_1}} r^{-\alpha_1+1-\delta} Q_1(x_1, x_2, x_3)\right. \\
&\quad \left. + \frac{x_2}{r^{\alpha_2}} r^{-\alpha_2+1-\delta} Q_2(x_1, x_2, x_3) + \frac{x_3}{r^{\alpha_3}} r^{-\alpha_3+1-\delta} Q_3(x_1, x_2, x_3)\right] \alpha_i \frac{x_i}{r^{\alpha_i}} \\
&= \left(\frac{\alpha_1 x_1^2}{r^{2\alpha_1}} + \frac{\alpha_2 x_2^2}{r^{2\alpha_2}} + \frac{\alpha_3 x_3^2}{r^{2\alpha_3}}\right) r^{-\alpha_i+1-\delta} Q_i(x_1, x_2, x_3) + \left[\frac{\alpha_1 x_1^2}{r^{\alpha_1}} r^{-\alpha_1+1-\delta} + \frac{\alpha_2 x_2^2}{r^{\alpha_2}} r^{-\alpha_2+1-\delta}\right. \\
&\quad \left. + \frac{\alpha_3 x_3}{r^{\alpha_3}} r^{-\alpha_3+1-\delta}\right] \frac{\alpha_i x_i}{r^{\alpha_i}} f(x_1, x_2, x_3) \\
&= \left(\frac{\alpha_1 x_1^2}{r^{2\alpha_1}} + \frac{\alpha_2 x_2^2}{r^{2\alpha_2}} + \frac{\alpha_3 x_3^2}{r^{2\alpha_3}}\right) r^{-\alpha_i+1-\delta} \cdot [Q_i(x_1, x_2, x_3) + \alpha_i x_i f(x_1, x_2, x_3)] \\
&= 0.
\end{aligned}$$

Therefore $g = (x_1/r^{\alpha_1}, x_2/r^{\alpha_2}, x_3/r^{\alpha_3})$ is a singular point of vector field $Q_T(u)$. $\square$

Theorem 3.2. *If $g = (g_1, g_2, g_3)$ is a singular point of the vector field $Q_T(u)$ on the sphere S^2, then there is a singular point E of the vector field $xF(x) + Q(x)$ on the invariant curve $L_{Og} = \{(\lambda^{\alpha_1} g_1, \lambda^{\alpha_2} g_2, \lambda^{\alpha_3} g_3) \mid (g_1, g_2, g_3) = g,\ \lambda \in R^+\}$, if and only if $R(g)G(g) < 0$. Moreover, we have the following conclusions for all $x_0 \in L_{Og}$:*

1. *if $R(g) > 0,\ G(g) > 0, \lim_{t\to+\infty} x(t, x_0) = (g, \infty)$;*
2. *if $R(g) < 0,\ G(g) < 0, \lim_{t\to+\infty} x(t, x_0) = O$;*
3. *if $R(g) > 0,\ G(g) < 0, \lim_{t\to+\infty} x(t, x_0) = E$;*
4. *if $R(g) < 0,\ G(g) > 0, \lim_{t\to+\infty} x(t) = O$ or (g, ∞).*

Proof. We need only to prove the sufficient condition that the vector field $xF(x) + Q(x)$ has a singular point $E(x_1, x_2, x_3) \in L_{Og}$ if $R(g)G(g) < 0$. The necessary condition is Theorem 3.1.

If $g(g_1, g_2, g_3)$ is a singular point of the vector field $Q_T(u)$ and $R(g)G(g) < 0$, then $\langle \overline{g}, g\rangle Q(g) - \langle g, Q(g)\rangle \overline{g} = 0$. For any given point $x_0 \in L_{Og}$ there is a $\lambda \in (0, \infty)$ such that $x_0 = (\lambda^{\alpha_1} g_1, \lambda^{\alpha_2} g_2, \lambda^{\alpha_3} g_3)$ and

$$\begin{aligned} x_0F(x_0) + Q(x_0) &= \lambda^{\alpha_i} g_i \alpha_i f(\lambda^{\alpha_1} g_1, \lambda^{\alpha_2} g_2, \lambda^{\alpha_3} g_3) + Q_i(\lambda^{\alpha_1} g_1, \lambda^{\alpha_2} g_2, \lambda^{\alpha_3} g_3) \\ &= \alpha_i g_i \lambda^{\alpha_i + m} f(g) + \lambda^{\alpha_i - 1 + \delta} G_i(g) \\ &= \alpha_i g_i \lambda^{\alpha_i + m} f(g) + \lambda^{\alpha_i - 1 + \delta} \frac{\alpha_i g_i R(g)}{\langle \overline{g}, g\rangle} \\ &= \alpha_i g_i \lambda^{\alpha_i - 1 + \delta} \left[\lambda^{m+1-\delta} f(g) + \frac{R(g)}{\langle \overline{g}, g\rangle}\right] \\ &= \alpha_i g_i \lambda^{\alpha_i - 1 + \delta} f(g) \left[\lambda^{m+1-\delta} + \frac{R(g)}{G(g)}\right]. \end{aligned} \tag{3.1}$$

Equation $\lambda^{m+1-\delta} + \frac{R(g)}{G(g)} = 0$ has only one positive root $\lambda_0 = \big[-R(g)/G(g)\big]^{\frac{1}{m+1-\delta}}$. Therefore, x_0 is the only singular point of $xF(x) + Q(x)$ on the invariant curve L_{Og}.

If $R(g) > 0, G(g) > 0$, by the equation (2.2a) we have

$$\begin{aligned} \frac{dr^{\delta-m-1}}{dt} &= (\delta - m - 1) r^{\delta-m-2} [r^{m+2-\delta} G(u) + rR(u)] \\ &= (\delta - m - 1) G(u) \left[1 + r^{\delta-m-1} \frac{R(u)}{G(u)}\right] \end{aligned} \tag{3.2}$$

Then, $\lim_{t\to+\infty} r^{\delta-m-1}(t) = 0,\ \lim_{t\to+\infty} r(t) = \infty$. Therefore, $\lim_{t\to+\infty} x(t, x_0) = (g, \infty)$.

Similarly, if $R(g) < 0, G(g) < 0$, we have $\lim_{t\to+\infty} r(t) = 0$, then $\lim_{t\to+\infty} x(t, x_0) = O$; if $R(g) > 0, G(g) < 0$, we have $\lim_{t\to+\infty} r(t) = r_0$, then $\lim_{t\to+\infty} x(t, x_0) = E$, where r_0 is the singular point of equation (3.2); if $R(g) < 0, G(g) > 0$, we have $\lim_{t\to-\infty} r(t) = r_0$, then $\lim_{t\to-\infty} x(t, x_0) = E$ (or $\lim_{t\to+\infty} x(t, x_0) = O$ or (g, ∞)). □

Remark 3.3. *We use (g, ∞) to denote a point at infinity along the invariant curve L_{Og}.*

Lemma 3.4 ([8]). *If γ is a trajectory of the vector field $Q_T(u)$ on the sphere S^2, then $S(\gamma)$ is an invariant quasi-cone of the vector field $xF(x) + Q(x)$.*

Theorem 3.5. *Let $\Omega_\gamma = g_1$, $A_\gamma = g_2$, $\gamma = \{u(t) \mid u(t) \in S^2,\ t \in (-\infty, +\infty)\}$.*

1. *If* $R(g_1) > 0, G(g_1) < 0$, *then there is a singular point* E_1 *of the vector field* $xF(x) + Q(x)$ *such that*
$$\lim_{t\to+\infty} x(t, x_0) = E_1, \quad \forall x_0 \in S(\gamma) - L_{Og_2}.$$
2. *If* $R(g_1) > 0, G(g_1) > 0$, *then* $\lim_{t\to+\infty} x(t, x_0) = (g, \infty), \forall x_0 \in S(\gamma) - L_{Og_2}$.
3. *If* $R(g_1) < 0, G(g_1) < 0$, *then* $\lim_{t\to+\infty} x(t, x_0) = O, \forall x_0 \in S(\gamma) - L_{Og_2}$.
4. *If* $R(g_1) < 0, G(g_1) > 0$, *then* $\lim_{t\to+\infty} x(t) = O$ *or* $(g, \infty)), \forall x_0 \in S(\gamma) - L_{Og_2}$.

Proof. (1) If $x_0 \in L_{Og_1}$, then, the existence E_1 as a singular point of vector field $xF(x) + Q(x)$ and $\lim_{t\to+\infty} x(t, x_0) = E_1, \forall x_0 \in S(\gamma) - L_{Og_1}$ are obvious by the result of Theorem 3.2.

If $x_0 \in S(\gamma) - (L_{Og_1} \cup L_{Og_2})$, let $u_0 = (x_{01}/r^{2\alpha_1}, x_{02}/r^{2\alpha_2}, x_{03}/r^{2\alpha_3})$ (where $r^{-2\alpha_1}x_{01}^2 + r^{-2\alpha_2}x_{02}^2 + r^{-2\alpha_3}x_{03}^2 = 1$), then $u_0 \in \gamma$ and $\lim_{t\to\infty} u(t, u_0) = g_1$. $R(u), G(u)$ are continuous functions of variables $u = (u_1, u_2, u_3)$. For all $\varepsilon > 0$, there is $T_1 = T_1(\varepsilon, u_0)$ such that for $t > T_1$

$$\begin{aligned} R(g_1) - \varepsilon < R(u(t, u_0)) < R(g_1) + \varepsilon, \\ G(g_1) - \varepsilon < G(u(t, u_0)) < G(g_1) + \varepsilon. \end{aligned} \tag{3.3}$$

By equation (3.2), we have

$$\frac{dr^{\delta-m-1}}{dt} = (\delta - m - 1)G(u)\left[1 + r^{\delta-m-1}\frac{R(u)}{G(u)}\right]$$

We construct the following equation:

$$\begin{aligned} \frac{dr_1^{\delta-m-1}}{dt} &= (\delta - m - 1)(G(g_1) + \varepsilon)\left[1 + r_1^{\delta-m-1}\frac{R(g_1) + \varepsilon}{G(g_1) + \varepsilon}\right], \\ \frac{dr_2^{\delta-m-1}}{dt} &= (\delta - m - 1)(G(g_1) - \varepsilon)\left[1 + r_2^{\delta-m-1}\frac{R(g_1) - \varepsilon}{G(g_1) - \varepsilon}\right]. \end{aligned}$$

By the comparison theorem of ordinary differential equations [4, 7] and inequalities (3.3), we have

$$r_1(t, r_0) < r(t, r_0) < r_2(t, r_0).$$

when $r_2(t_0) = r(t_0) = r_1(t_0)$ and $t_0 > T_1$. Since

$$\lim_{t\to\infty} r_1(t, r_0) = \left[-\frac{R(g_1) + \varepsilon}{G(g_1) + \varepsilon}\right]^{\frac{1}{m+1-\delta}},$$

$$\lim_{t\to\infty} r_2(t, r_0) = \left[-\frac{R(g_1) - \varepsilon}{G(g_1) - \varepsilon}\right]^{\frac{1}{m+1-\delta}},$$

and ε could be a sufficient small positive number, we could obtain

$$\lim_{t\to\infty} r(t, r_0) = [-R(u)/G(u)]^{1/(m+1-\delta)}, \quad \lim_{t\to\infty} x(t, x_0) = E_1.$$

The first part has been proved.

The proof of the remaining parts are similar to the first part, we omit it. □

Corollary 3.6. *Let* $\Omega_\gamma = g_1$, $A_\gamma = g_2$, $\gamma = \{u(t) \mid u(t) \in S^2,\ t \in (-\infty, +\infty)\}$. *If* $R(g_1) > 0$, $G(g_1) < 0$, $R(g_2) > 0$, $G(g_2) < 0$, *then there are two singular points* E_1 ($E_1 \in L_{Og_1}$), E_2 ($E_2 \in L_{Og_2}$) *and a unique trajectory* w^* *connected with saddles of* $xF(x) + Q(x)$ *such that* $\Omega_{w^*} = E_1$, $A_{w^*} = E_2$.

4 Classification of integral quasi-cones

For convenience of the following discussion, we first introduce some definitions and notations. Similar to Remark 3.3, we define $(\theta, \infty), (G, \infty)$ for a closed orbit, a graph of $xF(x) + Q(x)$ at infinity respectively, where $+\infty$ stands for $r \to +\infty$.

Definition 4.1. $S(\gamma)$ is a parabolic quasi-cone of the 1st kind if each $w \in S(\gamma)$ such that $\Omega_w = O,\ A_w = (g, +\infty)$, or $\Omega_w = (g, +\infty),\ A_w = O$; $S(\gamma)$ is a parabolic quasi-cone of the 2nd kind if each $w \in S(\gamma)$ such that $\Omega_w = O,\ A_w = (\theta, +\infty)$, or $\Omega_w = (\theta, +\infty),\ A_w = O$; $S(\gamma)$ is a parabolic quasi-cone of the 3rd kind if each $w \in S(\gamma)$ such that $\Omega_w = O,\ A_w = (G, +\infty)$, or $\Omega_w = (G, +\infty),\ A_w = O$.

Definition 4.2. $S(\gamma)$ is a hyperbolic quasi-cone of the 1st kind if each $w \in S(\gamma)$ such that $\Omega_w = (g_1, +\infty), A_w = (g_2, +\infty)$;

$S(\gamma)$ is a hyperbolic quasi-cone of the 2nd kind if each $w \in S(\gamma)$ such that $\Omega_w = (g, +\infty)$, $A_w = (\theta, +\infty)$, or $\Omega_w = (\theta, +\infty)$, $A_w = (g, +\infty)$;

$S(\gamma)$ is a hyperbolic quasi-cone of the 3rd kind if each $w \in S(\gamma)$ such that $\Omega_w = (\theta_1, +\infty)$, $A_w = (\theta_2, +\infty)$;

$S(\gamma)$ is a hyperbolic quasi-cone of the 4th kind if each $w \in S(\gamma)$ such that $\Omega_w = (g, +\infty)$, $A_w = (G, +\infty)$, or $\Omega_w = (G, +\infty)$, $A_w = (g, +\infty)$;

$S(\gamma)$ is a hyperbolic quasi-cone of the 5th kind if each $w \in S(\gamma)$ such that $\Omega_w = (\theta, +\infty)$, $A_w = (G, +\infty)$, or $\Omega_w = (G, +\infty)$, $A_w = (\theta, +\infty)$;

$S(\gamma)$ is a hyperbolic quasi-cone of the 6th kind if each $w \in S(\gamma)$ such that $\Omega_w = (G_1, +\infty)$, $A_w = (G_2, +\infty)$.

Definition 4.3. Let $S(\gamma)$ be a center-type quasi-cone.

$S(\gamma)$ is a quasi-cone of type P of the 1st kind if each $w \in S(\gamma)$ such that $\Omega_w = O$, $A_w = \theta^*$, or $\Omega_w = \theta^*$, $A_w = O$;

$S(\gamma)$ is a quasi-cone type P of the 2nd kind if each $w \in S(\gamma)$ such that $\Omega_w = (g, +\infty)$, $A_w = \theta^*$, or $\Omega_w = \theta^*$, $A_w = (g, +\infty)$;

$S(\gamma)$ is a quasi-cone type P of the 3rd kind if each $w \in S(\gamma)$ such that $\Omega_w = \theta_1^*$, $A_w = \theta_2^*$;

$S(\gamma)$ is a quasi-cone type P of the 4th kind if each $w \in S(\gamma)$ such that $\Omega_w = \theta^*$, $A_w = (\theta_1, +\infty)$, or $\Omega_w = (\theta_1, +\infty)$, $A_w = \theta^*$;

$S(\gamma)$ is a quasi-cone type P of the 5th kind if each $w \in S(\gamma)$ such that $\Omega_w = \theta^*$, $A_w = (G, +\infty)$, or $\Omega_w = (G, +\infty)$, $A_w = \theta^*$.

Definition 4.4. Let $S(\gamma)$ be a quasi-cone with singular point without origin and at infinity.

$S(\gamma)$ is a quasi-cone of type S of the 1st kind if each $w \in S(\gamma)$ such that $\Omega_w = O$, $A_w = E$, or $\Omega_w = E$, $A_w = O$;

$S(\gamma)$ is a quasi-cone of type S of the 2nd kind if each $w \in S(\gamma)$ such that $\Omega_w = (g, +\infty)$, $A_w = E$, or $\Omega_w = E$, $A_w = (g, +\infty)$;

$S(\gamma)$ is a quasi-cone of type S of the 3rd kind if each $w \in S(\gamma)$ such that $\Omega_w = E_1$, $A_w = E_2$;

$S(\gamma)$ is a quasi-cone of type S of the 4th kind if each $w \in S(\gamma)$ such that $\Omega_w = E$, $A_w = (\theta, +\infty)$, or $\Omega_w = (\theta, +\infty)$, $A_w = E$;

$S(\gamma)$ is a quasi-cone of type S of the 5th kind if each $w \in S(\gamma)$ such that $\Omega_w = E$, $A_w = (G, +\infty)$, or $\Omega_w = (G, +\infty)$, $A_w = E$.

Definition 4.5. $S(\gamma)$ is a P-S type quasi-cone if each $w \in S(\gamma)$ such that $\Omega_w = \theta^*$, $A_w = E$, or $\Omega_w = E$, $A_w = \theta^*$;

Theorem 4.6. *Let $\Omega_\gamma = g_1,\ A_\gamma = g_2, \gamma = \{u(t) : t \in (-\infty, +\infty)\}$. Then*
(1) $S(\gamma)$ is a quasi-cone of type S of the 1st or 2nd kind if $R(g_1) > 0,\ G(g_1) < 0,\ R(g_2) > 0,\ G(g_2) < 0$; or $R(g_1) < 0,\ G(g_1) > 0,\ R(g_2) < 0,\ G(g_2) > 0$ (Figure 4.1.1);
(2) $S(\gamma)$ is a quasi-cone of type S of the 3rd kind if $R(g_1) > 0, G(g_1) < 0, R(g_2) < 0, G(g_2) > 0$ (Figure 4.1.2);
(3) $S(\gamma)$ is a hyperbolic quasi-cone of the 1st kind or a parabolic quasi-cone of the 1st kind or an elliptic quasi-cone if $R(g_1) < 0, G(g_1) > 0, R(g_2) > 0, G(g_2) < 0$ (Figure 4.1.3);
(4) $S(\gamma)$ is a hyperbolic quasi-cone of the 1st kind or a parabolic quasi-cone of the 1st kind if $R(g_1) > 0, G(g_1) > 0, R(g_2) > 0, G(g_2) < 0$; or $R(g_1) < 0, G(g_1) > 0, R(g_2) < 0, G(g_2) < 0$ (Figure 4.1.4);
(5) $S(\gamma)$ is a quasi-cone of type S of the 2nd kind if $R(g_1) > 0, G(g_1) > 0, R(g_2) < 0, G(g_2) > 0$; or $R(g_1) > 0, G(g_1) < 0, R(g_2) < 0, G(g_2) < 0$ (Figure 4.1.5);
(6) $S(\gamma)$ is a parabolic quasi-cone of the 1st kind or an elliptic quasi-cone if $R(g_1) < 0, G(g_1) < 0, R(g_2) > 0, G(g_2) < 0$; or $R(g_1) < 0,\ G(g_1) > 0, R(g_2) > 0, G(g_2) > 0$ (Figure 4.1.6);
(7) $S(\gamma)$ is a quasi-cone of type S of the 1st kind if $R(g_1) < 0, G(g_1) < 0, R(g_2) < 0, G(g_2) > 0$; or $R(g_1) > 0, G(g_1) < 0, R(g_2) > 0, G(g_2) > 0$ (Figure 4.1.7);
(8) $S(\gamma)$ is a parabolic quasi-cone of the 1st kind if $R(g_1) > 0, G(g_1) > 0, R(g_2) > 0, G(g_2) > 0$; or $R(g_1) < 0, G(g_1) < 0, R(g_2) < 0, G(g_2) < 0$ (Figure 4.1.8);
(9) $S(\gamma)$ is a hyperbolic quasi-cone of the 1st kind if $R(g_1) > 0, G(g_1) > 0, R(g_2) < 0, G(g_2) < 0$ (Figure 4.1.9);
(10) $S(\gamma)$ is an elliptic quasi-cone if $R(g_1) < 0, G(g_1) < 0, R(g_2) > 0, G(g_2) > 0$ (Figure 4.1.10);

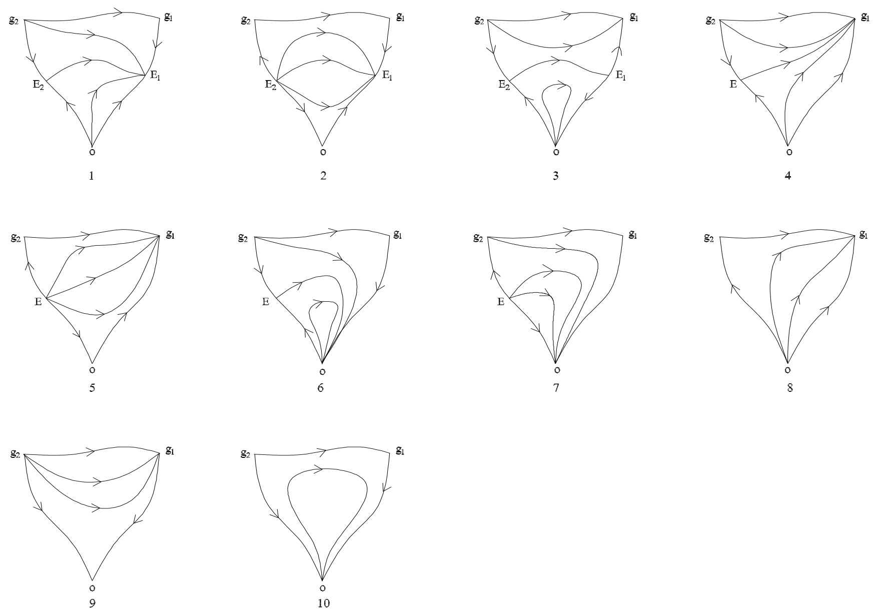

Figure 4.1: The classification of integral quasi-cones of $\Omega_\gamma = g_1, A_\gamma = g_2$

The proof of this theorem is similar to the proof of Theorem 3.5, we omit it.

Let $I(\theta) = \int_0^T R(\theta(s))\,ds, H(\theta) = \int_0^T G(\theta(s))\,ds$, θ is a closed orbit of $Q_T(u)$ on S^2.

Theorem 4.7. *Let $\Omega_\gamma = \theta, A_\gamma = g, \gamma = \{u(t) : t \in (-\infty, +\infty)\}$, then*
(1) $S(\gamma)$ is a quasi-cone of type S of the 1st or 2nd kind if $I(\theta) > 0, H(\theta) < 0, R(g) > 0, G(g) < 0$;

(2) $S(\gamma)$ is a quasi-cone of type S of the 3rd kind if $I(\theta) > 0, H(\theta) < 0, R(g) < 0, G(g) > 0$;
(3) $S(\gamma)$ is a hyperbolic quasi-cone of the 2nd kind or a parabolic quasi-cone of the 1st or 2nd kind or an elliptic quasi-cone if $I(\theta) < 0, H(\theta) > 0, R(g) > 0, G(g) < 0$;
(4) $S(\gamma)$ is a quasi-cone of type S of the 1st or 4th kind if $I(\theta) < 0, H(\theta) > 0, R(g) < 0, G(g) > 0$;
(5) $S(\gamma)$ is a hyperbolic quasi-cone of the 2nd kind or a parabolic quasi-cone of the 2nd kind if one of the following conditions holds:
(a) $I(\theta) > 0, H(\theta) > 0, R(g) > 0, G(g) < 0$;
(b) $I(\theta) = 0, \lim_{t\to+\infty} \int_0^t R(u(s))\,ds = +\infty, R(g) > 0, G(g) < 0$;
(6) $S(\gamma)$ is a quasi-cone of type S of the 4th kind if one of the following conditions holds:
(a) $I(\theta) > 0, H(\theta) > 0, R(g) < 0, G(g) > 0$;
(b) $I(\theta) = 0, \lim_{t\to+\infty} \int_0^t R(u(s))\,ds = +\infty, R(g) < 0, G(g) > 0$;
(7) $S(\gamma)$ is a parabolic quasi-cone of 1st kind or an elliptic quasi-cone if one of the following conditions holds:
(a) $I(\theta) < 0, H(\theta) < 0, R(g) > 0, G(g) < 0$;
(b) $I(\theta) = 0, \lim_{t\to+\infty} \int_0^t R(u(s))\,ds = -\infty, R(g) > 0, G(g) < 0$;
(8) $S(\gamma)$ is a quasi-cone of type S of the 1st kind if one of the following conditions holds:
(a) $I(\theta) < 0, H(\theta) < 0, R(g) < 0, G(g) > 0$;
(b) $I(\theta) > 0, H(\theta) < 0, R(g) > 0, G(g) > 0$;
(c) $I(\theta) = 0, \lim_{t\to+\infty} \int_0^t R(u(s))\,ds = -\infty, R(g) < 0, G(g) > 0$;
(9) $S(\gamma)$ is a parabolic quasi-cone of the 2nd kind if one of the following conditions holds:
(a) $I(\theta) > 0, H(\theta) > 0, R(g) > 0, G(g) > 0$;
(b) $I(\theta) = 0, \lim_{t\to+\infty} \int_0^t R(u(s))\,ds = +\infty, R(g) > 0, G(g) > 0$;
(10) $S(\gamma)$ is a hyperbolic quasi-cone of the 2nd kind if one of the following conditions holds:
(a) $I(\theta) > 0, H(\theta) > 0, R(g) < 0, G(g) < 0$;
(b) $I(\theta) = 0, \lim_{t\to+\infty} \int_0^t R(u(s))\,ds = +\infty, R(g) < 0, G(g) < 0$;
(11) $S(\gamma)$ is an elliptic quasi-cone if one of the following conditions holds:
(a) $I(\theta) < 0, H(\theta) < 0, R(g) > 0, G(g) > 0$;
(b) $I(\theta) = 0, \lim_{t\to+\infty} \int_0^t R(u(s))\,ds = -\infty, R(g) > 0, G(g) > 0$;
(12) $S(\gamma)$ is a parabolic quasi-cone of the 1st kind if one of the following conditions holds:
(a) $I(\theta) < 0, H(\theta) < 0, R(g) < 0, G(g) < 0$;
(b) $I(\theta) = 0, \lim_{t\to+\infty} \int_0^t R(u(s))\,ds = -\infty, R(g) < 0, G(g) < 0$;
(13) $S(\gamma)$ is a quasi-cone of type S of the 2nd kind if $I(\theta) > 0, H(\theta) < 0, R(g) < 0, G(g) < 0$;
(14) $S(\gamma)$ is a parabolic quasi-cone of the 2nd kind or an elliptic quasi-cone if $I(\theta) < 0, H(\theta) > 0, R(g) > 0, G(g) > 0$;
(15) $S(\gamma)$ is a parabolic quasi-cone of the 1st kind or a hyperbolic quasi-cone of the 2nd kind if $I(\theta) < 0, H(\theta) > 0, R(g) < 0, G(g) < 0$;
(16) $S(\gamma)$ is a quasi-cone of type P of the 1st kind if $I(\theta) = H(\theta) = 0, \lim_{t\to+\infty} \int_0^t R(u(s))\,ds \neq \pm\infty, R(g) > 0, G(g) > 0$;
(17) $S(\gamma)$ is a quasi-cone of type P of the 1st or 2nd kind if $I(\theta) = H(\theta) = 0, \lim_{t\to+\infty} \int_0^t R(u(s))\,ds \neq \pm\infty, R(g) > 0, G(g) < 0$;
(18) $S(\gamma)$ is a quasi-cone of type $P - S$ if $I(\theta) = H(\theta) = 0$, $\lim_{t\to+\infty} \int_0^t R(u(s))\,ds \neq \pm\infty$, $R(g) < 0, G(g) > 0$;
(19) $S(\gamma)$ is a quasi-cone of type P of the 2nd kind if $I(\theta) = H(\theta) = 0, \lim_{t\to+\infty} \int_0^t R(u(s))\,ds \neq \pm\infty$, $R(g) < 0, G(g) < 0$.

Theorem 4.8. *Let $\Omega_\gamma = \theta_1, A_\gamma = \theta_2, \gamma = \{u(t) : t \in (-\infty, +\infty)\}$, then*
(1) $S(\gamma)$ is a quasi-cone of type S of the 1st or 4th kind if one of the following conditions holds:
(a) $I(\theta_1) > 0, H(\theta_1) < 0, I(\theta_2) > 0, H(\theta_2) < 0$;

(b) $I(\theta_1) < 0, H(\theta_1) > 0, I(\theta_2) < 0, H(\theta_2) > 0$;

(2) $S(\gamma)$ *is a quasi-cone of type S of the 3rd kind if* $I(\theta_1) > 0, H(\theta_1) < 0, I(\theta_2) < 0, H(\theta_2) > 0$;

(3) $S(\gamma)$ *is a hyperbolic quasi-cone of the 3rd kind or a parabolic quasi-cone of the 2nd kind or an elliptic quasi-cone if* $I(\theta_1) <, H(\theta_1) > 0, I(\theta_2) < 0, H(\theta_2) > 0$;

(4) $S(\gamma)$ *is parabolic quasi-cone of the 2nd kind or a hyperbolic quasi-cone of the 3rd kind if one of the following conditions holds:*

(a) $I(\theta_1) > 0, H(\theta_1) > 0, I(\theta_2) > 0, H(\theta_2) < 0$;

(b) $I(\theta_1) < 0, H(\theta_1) < 0, I(\theta_2) < 0, H(\theta_2) > 0$;

(c) $I(\theta_1) = 0, \lim_{t\to+\infty} \int_0^t R(u(s))\, ds = +\infty, I(\theta_2) > 0, H(\theta_2) < 0$;

(d) $I(\theta_1) < 0, H(\theta_1) > 0, I(\theta_2) = 0, \lim_{t\to-\infty} \int_0^t R(u(s))\, ds = +\infty$;

(5) $S(\gamma)$ *is a quasi-cone of type S of the 4th kind if one of the following conditions holds:*

(a) $I(\theta_1) > 0, H(\theta_1) > 0, I(\theta_2) < 0, H(\theta_2) > 0$;

(b) $I(\theta_1) > 0, H(\theta_1) < 0, I(\theta_2) < 0, H(\theta_2) < 0$;

(c) $I(\theta_1) = 0, \lim_{t\to+\infty} \int_0^t R(u(s))\, ds = +\infty, I(\theta_2) < 0, H(\theta_2) > 0$;

(d) $I(\theta_1) > 0, H(\theta_1) < 0, I(\theta_2) = 0, \lim_{t\to-\infty} \int_0^t R(u(s))\, ds = +\infty$;

(6) $S(\gamma)$ *is a parabolic quasi-cone of the 2nd kind or an elliptic quasi-cone if one of the following conditions holds:*

(a) $I(\theta_1) < 0, H(\theta_1) < 0, I(\theta_2) > 0, H(\theta_2) < 0$;

(b) $I(\theta_1) < 0, H(\theta_1) > 0, I(\theta_2) > 0, H(\theta_2) > 0$;

(c) $I(\theta_1) = 0, \lim_{t\to+\infty} \int_0^t R(u(s))\, ds = -\infty, I(\theta_2) > 0, H(\theta_2) < 0$;

(d) $I(\theta_1) < 0, H(\theta_1) > 0, I(\theta_2) = 0, \lim_{t\to-\infty} \int_0^t R(u(s))\, ds = -\infty$;

(7) $S(\gamma)$ *is a quasi-cone of type S of the 1st kind if one of the following conditions holds:*

(a) $I(\theta_1) < 0, H(\theta_1) < 0, I(\theta_2) < 0, H(\theta_2) > 0$;

(b) $I(\theta_1) > 0, H(\theta_1) < 0, I(\theta_2) > 0, H(\theta_2) > 0$;

(c) $I(\theta_1) = 0, \lim_{t\to+\infty} \int_0^t R(u(s))\, ds = -\infty, I(\theta_2) < 0, H(\theta_2) > 0$;

(d) $I(\theta_1) > 0, H(\theta_1) < 0, I(\theta_2) = 0, \lim_{t\to-\infty} \int_0^t R(u(s))\, ds = -\infty$;

(8) $S(\gamma)$ *is a parabolic quasi-cone of the 2nd kind if one of the following conditions holds:*

(a) $I(\theta_1) > 0, H(\theta_1) > 0, I(\theta_2) > 0, H(\theta_2) > 0$;

(b) $I(\theta_1) < 0, H(\theta_1) < 0, I(\theta_2) < 0, H(\theta_2) < 0$;

(c) $I(\theta_1) = 0, \lim_{t\to+\infty} \int_0^t R(u(s))\, ds = +\infty, I(\theta_2) > 0, H(\theta_2) > 0$;

(d) $I(\theta_1) = 0, \lim_{t\to+\infty} \int_0^t R(u(s))\, ds = -\infty, I(\theta_2) < 0, H(\theta_2) < 0$;

(e) $I(\theta_1) > 0, H(\theta_1) > 0, I(\theta_2) = 0, \lim_{t\to-\infty} \int_0^t R(u(s))\, ds = -\infty$;

(f) $I(\theta_1) < 0, H(\theta_1) < 0, I(\theta_2) = 0, \lim_{t\to-\infty} \int_0^t R(u(s))\, ds = +\infty$;

(g) $I(\theta_1) = 0, I(\theta_2) = 0, \lim_{t\to+\infty} \int_0^t R(u(s))\, ds = +\infty, \lim_{t\to-\infty} \int_0^t R(u(s))\, ds = -\infty$;

(i) $I(\theta_1) = 0, I(\theta_2) = 0, \lim_{t\to+\infty} \int_0^t R(u(s))\, ds = -\infty, \lim_{t\to-\infty} \int_0^t R(u(s))\, ds = +\infty$;

(9) $S(\gamma)$ *is a hyperbolic quasi-cone of the 3rd kind if one of the following conditions holds:*

(a) $I(\theta_1) > 0, H(\theta_1) > 0, I(\theta_2) < 0, H(\theta_2) < 0$;

(b) $I(\theta_1) = 0, \lim_{t\to+\infty} \int_0^t R(u(s))\, ds = +\infty, I(\theta_2) < 0, H(\theta_2) < 0$;

(c) $I(\theta_1) = 0, I(\theta_2) = 0, \lim_{t\to\pm\infty} \int_0^t R(u(s))\, ds = +\infty$;

(d) $I(\theta_1) > 0, H(\theta_1) > 0, I(\theta_2) = 0, \lim_{t\to-\infty} \int_0^t R(u(s))\, ds = +\infty$;

(10) $S(\gamma)$ *is an elliptic quasi-cone if one of the following conditions holds:*

(a) $I(\theta_1) < 0, H(\theta_1) < 0, I(\theta_2) > 0, H(\theta_2) > 0$;

(b) $I(\theta_1) = 0, \lim_{t\to+\infty} \int_0^t R(u(s))\, ds = -\infty, I(\theta_2) > 0, H(\theta_2) > 0$;

(c) $I(\theta_1) = 0, I(\theta_2) = 0, \lim_{t\to\pm\infty} \int_0^t R(u(s))\, ds = -\infty$;

(d) $I(\theta_1) < 0, H(\theta_1) < 0, I(\theta_2) = 0, \lim_{t\to-\infty} \int_0^t R(u(s))\, ds = -\infty$;

(11) $S(\gamma)$ is a quasi-cone of type P of the 1st kind if one of the following conditions holds:
(a) $I(\theta_1) = H(\theta_1) = 0, \lim_{t\to+\infty} \int_0^t R(u(s))\,ds \neq \pm\infty, I(\theta_2) > 0, H(\theta_2) > 0$;
(b) $I(\theta_1) = 0, I(\theta_2) = 0, \lim_{t\to+\infty} \int_0^t R(u(s))\,ds = -\infty, \lim_{t\to-\infty} \int_0^t R(u(s))\,ds \neq \pm\infty$;
(c) $I(\theta_1) = 0, I(\theta_2) = 0, \lim_{t\to+\infty} \int_0^t R(u(s))\,ds \neq \pm\infty, \lim_{t\to-\infty} \int_0^t R(u(s))\,ds = -\infty$;
(d) $I(\theta_1) < 0, H(\theta_1) < 0, I(\theta_2) = H(\theta_2) = 0, \lim_{t\to-\infty} \int_0^t R(u(s))\,ds \neq \pm\infty$;
(12) $S(\gamma)$ is a quasi-cone of type P of the 1st or 4th kind if one of the following conditions holds:
(a) $I(\theta_1) = H(\theta_1) = 0, \lim_{t\to+\infty} \int_0^t R(u(s))\,ds \neq \pm\infty, I(\theta_2) > 0, H(\theta_2) < 0$;
(b) $I(\theta_1) = 0, I(\theta_2) = 0, \lim_{t\to+\infty} \int_0^t R(u(s))\,ds = +\infty, \lim_{t\to-\infty} \int_0^t R(u(s))\,ds \neq \pm\infty$;
(c) $I(\theta_1) = 0, I(\theta_2) = 0, \lim_{t\to+\infty} \int_0^t R(u(s))\,ds \neq \pm\infty, \lim_{t\to-\infty} \int_0^t R(u(s))\,ds = +\infty$;
(d) $I(\theta_1) < 0, H(\theta_1) > 0, I(\theta_2) = H(\theta_2) = 0, \lim_{t\to-\infty} \int_0^t R(u(s))\,ds \neq \pm\infty$;
(13) $S(\gamma)$ is a quasi-cone of type $P - S$ if one of the following conditions holds:
(a) $I(\theta_1) = H(\theta_1) = 0, \lim_{t\to+\infty} \int_0^t R(u(s))\,ds \neq \pm\infty, I(\theta_2) < 0, H(\theta_2) > 0$;
(b) $I(\theta_1) > 0, H(\theta_1) < 0, I(\theta_2) = H(\theta_2) = 0, \lim_{t\to-\infty} \int_0^t R(u(s))\,ds \neq \pm\infty$;
(14) $S(\gamma)$ is a quasi-cone of type P of the 4th kind if one of the following conditions holds:
(a) $I(\theta_1) = H(\theta_1) = 0, \lim_{t\to+\infty} \int_0^t R(u(s))\,ds \neq \pm\infty, I(\theta_2) < 0, H(\theta_2) < 0$;
(b) $I(\theta_1) > 0, H(\theta_1) > 0, I(\theta_2) = H(\theta_2) = 0, \lim_{t\to-\infty} \int_0^t R(u(s))\,ds \neq \pm\infty$;
(15) $S(\gamma)$ is a quasi-cone of type P of the 3rd kind if $I(\theta_1) = H(\theta_1) = I(\theta_2) = H(\theta_2) = 0$, $\lim_{t\to\pm\infty} \int_0^t R(u(s))\,ds \neq \pm\infty, I(\theta_2) < 0, H(\theta_2) < 0$.

Theorem 4.9. *Let $\Omega_\gamma = G, A_\gamma = g, \gamma = \{u(t) : t \in (-\infty, +\infty)\}$, then*
(1) $S(\gamma)$ is a quasi-cone of type S of the 1st or 2nd kind if all $g_i \in G$ such that $R(g_i) > 0$, $G(g_i) < 0, R(g) > 0, G(g) < 0$;
(2) $S(\gamma)$ is a quasi-cone of type S of the 3rd kind if all $g_i \in G$ such that $R(g_i) > 0$, $G(g_i) < 0$, $R(g) < 0$, $G(g) > 0$;
(3) $S(\gamma)$ is a hyperbolic quasi-cone of the 4th kind or a parabolic quasi-cone of the 1st or 3rd kind or an elliptic quasi-cone if all $g_i \in G$ such that $R(g_i) < 0, G(g_i) > 0, R(g) > 0, G(g) < 0$;
(4) $S(\gamma)$ is a quasi-cone of type S of the 1st or 5th kind if all $g_i \in G$ such that $R(g_i) < 0$, $G(g_i) > 0, R(g) < 0, G(g) > 0$;
(5) $S(\gamma)$ is a parabolic quasi-cone of the 3rd kind or a hyperbolic quasi-cone of the 4th kind if all $g_i \in G$ such that $R(g_i) > 0, G(g_i) > 0, R(g) > 0, G(g) < 0$; or $\lim_{t\to+\infty} \int_0^t R(u(s))\,ds = +\infty$, $R(g) > 0, G(g) < 0$;
(6) $S(\gamma)$ is a quasi-cone of type S of the 5th kind if all $g_i \in G$ such that $R(g_i) > 0, G(g_i) > 0$, $R(g) < 0, G(g) > 0$; or $\lim_{t\to+\infty} \int_0^t R(u(s))\,ds = +\infty, R(g) < 0, G(g) > 0$;
(7) $S(\gamma)$ is a parabolic quasi-cone of the 1st kind or an elliptic quasi-cone if all $g_i \in G$ such that $R(g_i) < 0$, $G(g_i) < 0$, $R(g) > 0$, $G(g) < 0$; or $\lim_{t\to+\infty} \int_0^t R(u(s))\,ds = -\infty, R(g) > 0$, $G(g) < 0$;
(8) $S(\gamma)$ is a quasi-cone of type S of the 1st kind if all $g_i \in G$ such that $R(g_i) < 0, G(g_i) < 0$, $R(g) < 0, G(g) > 0$; or $R(g_i) > 0, G(g_i) < 0, R(g) > 0, G(g) > 0$; or $\lim_{t\to+\infty} \int_0^t R(u(s))\,ds = -\infty, R(g) < 0, G(g) > 0$;
(9) $S(\gamma)$ is a quasi-cone of type S of the 2nd kind if all $g_i \in G$ such that $R(g_i) > 0$, $G(g_i) < 0$, $R(g) < 0, G(g) < 0$;
(10) $S(\gamma)$ is a parabolic quasi-cone of the 4th kind or an elliptic quasi-cone if all $g_i \in G$ such that $R(g_i) < 0, G(g_i) > 0, R(g) > 0, G(g) > 0$;
(11) $S(\gamma)$ is a parabolic quasi-cone of the 1st kind or a hyperbolic quasi-cone of the 4th kind if all $g_i \in G$ such that $R(g_i) < 0$, $G(g_i) > 0$, $R(g) < 0$, $G(g) < 0$;
(12) $S(\gamma)$ is a parabolic quasi-cone of the 3rd kind if all $g_i \in G$ such that $R(g_i) > 0$, $G(g_i) > 0$, $R(g) > 0, G(g) > 0$; or $\lim_{t\to+\infty} \int_0^t R(u(s))\,ds = +\infty, R(g) > 0, G(g) > 0$;

(13) $S(\gamma)$ *is a hyperbolic quasi-cone of the 4th kind if all* $g_i \in G$ *such that* $R(g_i) > 0$, $G(g_i) > 0$, $R(g) < 0$, $G(g) < 0$; *or* $\lim_{t\to+\infty} \int_0^t R(u(s))\,ds = +\infty, R(g) < 0, G(g) < 0$;

(14) $S(\gamma)$ *is an elliptic quasi-cone if all* $g_i \in G$ *such that* $R(g_i) < 0, G(g_i) < 0, R(g) > 0, G(g) > 0$; *or* $\lim_{t\to+\infty} \int_0^t R(u(s))\,ds = -\infty, R(g) > 0, G(g) > 0$;

(15) $S(\gamma)$ *is a parabolic quasi-cone of the 1st kind if all* $g_i \in G$ *such that* $R(g_i) < 0$, $G(g_i) < 0$, $R(g) < 0, G(g) < 0$; *or* $\lim_{t\to+\infty} \int_0^t R(u(s))\,ds = -\infty, R(g) < 0, G(g) < 0$.

Theorem 4.10. *Let* $\Omega_\gamma = G, A_\gamma = \theta, \gamma = \{u(t) : t \in (-\infty, +\infty)$, *then*

(1) $S(\gamma)$ *is a quasi-cone of type S of the 1st or 4th kind if all* $g_i \in G$ *such that* $R(g_i) > 0, G(g_i) < 0$, $I(\theta) > 0, H(\theta) < 0$;

(2) $S(\gamma)$ *is a quasi-cone of type S of the 3rd kind if all* $g_i \in G$ *such that* $R(g_i) > 0, G(g_i) < 0$, $I(\theta) < 0, H(\theta) > 0$;

(3) $S(\gamma)$ *is a hyperbolic quasi-cone of the 5th kind or a parabolic quasi-cone of the 2nd or 3rd kind or an elliptic quasi-cone if all* $g_i \in G$ *such that* $R(g_i) < 0, G(g_i) > 0, I(\theta) > 0, H(\theta) < 0$;

(4) $S(\gamma)$ *is a quasi-cone of type S of the 1st or 5th kind if all* $g_i \in G$ *such that* $R(g_i) < 0, G(g_i) > 0$, $I(\theta) < 0, H(\theta) > 0$;

(5) $S(\gamma)$ *is a parabolic quasi-cone of the 3rd kind or a hyperbolic quasi-cone of the 5th kind if one of the following conditions holds:*
(a) all $g_i \in G$ *such that* $R(g_i) > 0, G(g_i) > 0, I(\theta) > 0, H(\theta) < 0$;
(b) $\lim_{t\to+\infty} \int_0^t R(u(s))\,ds = +\infty, I(\theta) > 0, H(\theta) < 0$;

(6) $S(\gamma)$ *is a quasi-cone of type S of the 5th kind if one of the following conditions holds:*
(a) all $g_i \in G$ *such that* $R(g_i) > 0, G(g_i) > 0, I(\theta) < 0, H(\theta) > 0$;
(b) $\lim_{t\to+\infty} \int_0^t R(u(s))\,ds = +\infty, I(\theta) < 0, H(\theta) > 0$;

(7) $S(\gamma)$ *is a parabolic quasi-cone of the 2nd kind or an elliptic quasi-cone if one of the following conditions holds:*
(a) all $g_i \in G$ *such that* $R(g_i) < 0, G(g_i) < 0, I(\theta) > 0, H(\theta) < 0$;
(b) $\lim_{t\to+\infty} \int_0^t R(u(s))\,ds = -\infty, I(\theta) > 0, H(\theta) < 0$;

(8) $S(\gamma)$ *is a quasi-cone of type S of the 1st kind if one of the following conditions holds:*
(a) all $g_i \in G$ *such that* $R(g_i) < 0, G(g_i) < 0, I(\theta) < 0, H(\theta) > 0$;
(b) all $g_i \in G$ *such that* $R(g_i) > 0, G(g_i) < 0, I(\theta) > 0, H(\theta) > 0$;
(c) all $g_i \in G$ *such that* $R(g_i) > 0, I(\theta) = 0, H(\theta) \neq 0, \lim_{t\to-\infty} \int_0^t R(u(s))\,ds = -\infty$;
(d) $\lim_{t\to+\infty} \int_0^t R(u(s))\,ds = -\infty, I(\theta) < 0, H(\theta) > 0$;

(9) $S(\gamma)$ *is a quasi-cone of type S of the 4th kind if one of the following conditions holds:*
(a) all $g_i \in G$ *such that* $R(g_i) > 0, G(g_i) < 0, I(\theta) < 0, H(\theta) < 0$;
(b) all $g_i \in G$ *such that* $R(g_i) > 0, I(\theta) = 0, H(\theta) \neq 0, \lim_{t\to-\infty} \int_0^t R(u(s))\,ds = +\infty$;

(10) $S(\gamma)$ *is a parabolic quasi-cone of the 3rd kind or an elliptic quasi-cone if one of the following conditions holds:*
(a) all $g_i \in G$ *such that* $R(g_i) < 0, G(g_i) > 0, I(\theta) > 0, H(\theta) > 0$;
(b) all $g_i \in G$ *such that* $R(g_i) < 0, I(\theta) = 0, H(\theta) \neq 0, \lim_{t\to-\infty} \int_0^t R(u(s))\,ds = -\infty$;

(11) $S(\gamma)$ *is a parabolic quasi-cone of the 2nd kind or a hyperbolic quasi-cone of the 5th kind if one of the following conditions holds:*
(a) all $g_i \in G$ *such that* $R(g_i) < 0, G(g_i) > 0, I(\theta) < 0, H(\theta) < 0$;
(b) all $g_i \in G$ *such that* $R(g_i) < 0, I(\theta) = 0, H(\theta) \neq 0, \lim_{t\to-\infty} \int_0^t R(u(s))\,ds = +\infty$;

(12) $S(\gamma)$ *is a parabolic quasi-cone of the 3rd kind if one of the following conditions holds:*
(a) all $g_i \in G$ *such that* $R(g_i) > 0, G(g_i) > 0, I(\theta) > 0, H(\theta) > 0$;
(b) all $g_i \in G$ *such that* $R(g_i) > 0, I(\theta) = 0, H(\theta) \neq 0, \lim_{t\to-\infty} \int_0^t R(u(s))\,ds = -\infty$;
(c) $\lim_{t\to+\infty} \int_0^t R(u(s))\,ds = +\infty, I(\theta) > 0, H(\theta) > 0$;

(d) $\lim_{t\to+\infty}\int_0^t R(u(s))\,ds = +\infty, I(\theta) = 0, \lim_{t\to-\infty}\int_0^t R(u(s))\,ds = -\infty$;

(13) $S(\gamma)$ is a hyperbolic quasi-cone of the 4th kind if one of the following conditions holds:
(a) all $g_i \in G$ such that $R(g_i) > 0, G(g_i) > 0, I(\theta) < 0, H(\theta) < 0$;
(b) all $g_i \in G$ such that $R(g_i) > 0, I(\theta) = 0, H(\theta) \neq 0, \lim_{t\to-\infty}\int_0^t R(u(s))\,ds = +\infty$;
(c) $\lim_{t\to+\infty}\int_0^t R(u(s))\,ds = +\infty, I(\theta) < 0, H(\theta) < 0$;
(d) $\lim_{t\to+\infty}\int_0^t R(u(s))\,ds = +\infty, I(\theta) = 0, \lim_{t\to-\infty}\int_0^t R(u(s))\,ds = +\infty$;

(14) $S(\gamma)$ is an elliptic quasi-cone if one of the following conditions holds:
(a) all $g_i \in G$ such that $R(g_i) < 0, G(g_i) < 0, I(\theta) > 0, H(\theta) > 0$;
(b) all $g_i \in G$ such that $R(g_i) < 0, I(\theta) = 0, H(\theta) \neq 0, \lim_{t\to-\infty}\int_0^t R(u(s))\,ds = -\infty$;
(c) $\lim_{t\to+\infty}\int_0^t R(u(s))\,ds = -\infty, I(\theta) > 0, H(\theta) > 0$;
(d) $\lim_{t\to+\infty}\int_0^t R(u(s))\,ds = -\infty, I(\theta) = 0, \lim_{t\to-\infty}\int_0^t R(u(s))\,ds = -\infty$;

(15) $S(\gamma)$ is a parabolic quasi-cone of the 2nd kind if one of the following conditions holds:
(a) all $g_i \in G$ such that $R(g_i) < 0, G(g_i) < 0, I(\theta) < 0, H(\theta) < 0$;
(b) all $g_i \in G$ such that $R(g_i) < 0, I(\theta) = 0, H(\theta) \neq 0, \lim_{t\to-\infty}\int_0^t R(u(s))\,ds = +\infty$;
(c) $\lim_{t\to+\infty}\int_0^t R(u(s))\,ds = -\infty, I(\theta) < 0, H(\theta) < 0$;
(d) $\lim_{t\to+\infty}\int_0^t R(u(s))\,ds = -\infty, I(\theta) = 0, \lim_{t\to-\infty}\int_0^t R(u(s))\,ds = +\infty$;

(16) $S(\gamma)$ is a quasi-cone of type P of the 5th kind if one of the following conditions holds:
(a) all $g_i \in G$ such that $R(g_i) > 0, G(g_i) > 0, I(\theta) = H(\theta) = 0, \lim_{t\to-\infty}\int_0^t R(u(s))\,ds \neq \pm\infty$;
(b) $\lim_{t\to+\infty}\int_0^t R(u(s))\,ds = +\infty, I(\theta) = 0 = H(\theta) = 0, \lim_{t\to-\infty}\int_0^t R(u(s))\,ds \neq \pm\infty$;

(17) $S(\gamma)$ is a quasi-cone of type $P - S$ if all $g_i \in G$ such that $R(g_i) > 0, G(g_i) < 0, I(\theta) = H(\theta) = 0$, $\lim_{t\to-\infty}\int_0^t R(u(s))\,ds \neq \pm\infty$;

(18) $S(\gamma)$ is a quasi-cone of type P of the 1st or 5th kind if all $g_i \in G$ such that $R(g_i) < 0, G(g_i) > 0$, $I(\theta) = H(\theta) = 0, \lim_{t\to-\infty}\int_0^t R(u(s))\,ds \neq \pm\infty$;

(19) $S(\gamma)$ is a quasi-cone of type P of the 1st kind if one of the following conditions holds:
(a) all $g_i \in G$ such that $R(g_i) < 0, G(g_i) < 0, I(\theta) = H(\theta) = 0, \lim_{t\to-\infty}\int_0^t R(u(s))\,ds \neq \pm\infty$;
(b) $\lim_{t\to+\infty}\int_0^t R(u(s))\,ds = -\infty, I(\theta) = H(\theta) = 0, \lim_{t\to-\infty}\int_0^t R(u(s))\,ds \neq \pm\infty$;

Theorem 4.11. *Let $\Omega_\gamma = G_1, A_\gamma = G_2, \gamma = \{u(t) : t \in (-\infty, +\infty)\}$, then*

(1) $S(\gamma)$ is a quasi-cone of type S of the 1st or 5th kind if all $g_i \in G_1, g_j \in G_2$ such that $R(g_i) > 0$, $G(g_i) < 0, R(g_j) > 0, G(g_j) < 0$;

(2) $S(\gamma)$ is a quasi-cone of type S of the 3rd kind if all $g_i \in G_1, g_j \in G_2$ such that $R(g_i) > 0, G(g_i) < 0$, $R(g_j) < 0, G(g_j) > 0$;

(3) $S(\gamma)$ is a hyperbolic quasi-cone of the 6th kind or a parabolic quasi-cone of the 3rd kind or an elliptic quasi-cone if all $g_i \in G_1, g_j \in G_2$ such that $R(g_i) < 0, G(g_i) > 0, R(g_j) > 0, G(g_j) < 0$;

(4) $S(\gamma)$ is a quasi-cone of type S of the 1st or 5th kind if all $g_i \in G_1, g_j \in G_2$ such that $R(g_i) < 0$, $G(g_i) > 0, R(g_j) < 0, G(g_j) > 0$;

(5) $S(\gamma)$ is a parabolic quasi-cone of the 3rd kind or a hyperbolic quasi-cone of the 6th kind if all $g_i \in G_1, g_j \in G_2$ such that $R(g_i) > 0, G(g_i) > 0, R(g) > 0, G(g) < 0$;

(6) $S(\gamma)$ is a quasi-cone of type S of the 5th kind if $R(g_i) > 0, G(g_i) > 0, R(g_j) < 0, G(g_j) > 0$;

(7) $S(\gamma)$ is a parabolic quasi-cone of the 3rd kind or an elliptic quasi-cone if all $g_i \in G_1, g_j \in G_2$ such that $R(g_i) < 0, G(g_i) < 0, R(g_j) > 0, G(g_j) < 0$;

(8) $S(\gamma)$ is a quasi-cone of type S of the 1st kind if one of the following conditions holds:
(a) all $g_i \in G_1, g_j \in G_2$ such that $R(g_i) < 0, G(g_i) < 0, R(g) < 0, G(g) > 0$;
(b) all $g_i \in G_1, g_j \in G_2$ such that $R(g_i) > 0, G(g_i) < 0, R(g_j) > 0, G(g_j) > 0$;

(9) $S(\gamma)$ is a quasi-cone of type S of the 5th kind if all $g_i \in G_1, g_j \in G_2$ such that $R(g_i) > 0, G(g_i) < 0$, $R(g_j) < 0, G(g_j) < 0$;

(10) $S(\gamma)$ *is a parabolic quasi-cone of the 3rd kind or an elliptic quasi-cone if all* $g_i \in G_1, g_j \in G_2$ *such that* $R(g_i) < 0, G(g_i) > 0, R(g_j) > 0, G(g_j) > 0$;

(11) $S(\gamma)$ *is a parabolic quasi-cone of the 3rd kind or a hyperbolic quasi-cone of the 6th kind if all* $g_i \in G_1, g_j \in G_2$ *such that* $R(g_i) < 0, G(g_i) > 0, R(g_j) < 0, G(g_j) < 0$;

(12) $S(\gamma)$ *is a parabolic quasi-cone of the 3rd kind if one of the following conditions holds:*

(a) all $g_i \in G_1, g_j \in G_2$ *such that* $R(g_i) > 0, G(g_i) > 0, R(g_j) > 0, G(g_j) > 0$;

(b) all $g_i \in G_1, g_j \in G_2$ *such that* $R(g_i) < 0, G(g_i) < 0, R(g_j) < 0, G(g_j) < 0$;

(c) $\lim_{t\to+\infty} \int_0^t R(u(s))\,ds = +\infty, \lim_{t\to-\infty} \int_0^t R(u(s))\,ds = -\infty$;

(d) $\lim_{t\to+\infty} \int_0^t R(u(s))\,ds = -\infty, \lim_{t\to-\infty} \int_0^t R(u(s))\,ds = +\infty$;

(13) $S(\gamma)$ *is a hyperbolic quasi-cone of the 6th kind if one of the following conditions holds:*

(a) all $g_i \in G_1, g_j \in G_2$ *such that* $R(g_i) > 0, G(g_i) > 0, R(g_j) < 0, G(g_j) < 0$;

(b) $\lim_{t\to+\infty} \int_0^t R(u(s))\,ds = +\infty, \lim_{t\to-\infty} \int_0^t R(u(s))\,ds = +\infty$;

(14) $S(\gamma)$ *is an elliptic quasi-cone if one of the following conditions holds:*

(a) $g_i \in G_1, g_j \in G_2$ *such that* $R(g_i) < 0, G(g_i) < 0, R(g_j) > 0, G(g_j) > 0$;

(b) $\lim_{t\to+\infty} \int_0^t R(u(s))\,ds = -\infty, \lim_{t\to-\infty} \int_0^t R(u(s))\,ds = -\infty$.

Theorem 4.12. *If the tangent vector field* $Q_T(u)$ *has a closed orbit* $\theta = \theta(s)$ *with period* T *on the sphere* S^2 *and* $\int_0^T R(\theta(s))\,ds \cdot \int_0^T e^{(\delta-m-1)\int_\tau^T R(\theta(s))\,ds} G(\theta(\tau))\,d\tau < 0$, *then the vector field* $xF(x) + Q(x)$ *has a closed orbit* $\theta^\star(\theta^\star \in C(\theta))$ *in* $\mathbb{R}^3$. *Furthermore, the closed orbit* $\theta^\star$ *is an attractor for other trajectories on* $S(\theta)$ *if* $\int_0^T R(\theta(s))\,ds > 0$.

Proof. By the discussion in Section 2 we know that the flows of the vector field $xF(x) + Q(x)$ in $\mathbb{R}^3$ are topologically equivalent to the flows of systems (2.2), and system (2.2b) has a closed orbit $\theta = \theta(s)$ with period T. If system (2.2a) has only one periodic solution $r(t, r_0^\star)$ ($r_0^\star > 0$ is the initial value) with period T on the closed surface $S(\theta)$, then there is a closed orbit $\theta^\star$ of the vector field $xF(x) + Q(x)$ in $\mathbb{R}^3$. As a matter of fact, the general solution of system (2.2a) on the closed surface $S(\theta)$ is

$$r(t, r_0) = e^{\int_0^t R(\theta(s))\,ds} \left[r_0^{\delta-m-1} + (\delta - m - 1) \int_0^t e^{-(\delta-m-1)\int_0^\tau R(\theta(s))\,ds} G(\theta(\tau))\,d\tau \right]^{\frac{1}{\delta-m-1}}.$$

Then,

$$\begin{aligned} r(t+T, r_0) = &\left[(r_0 e^{\int_0^T R(\theta(s))\,ds})^{\delta-m-1} + (\delta - m - 1) \int_0^T e^{(\delta-m-1)\int_\tau^T R(\theta(s))\,ds} G(\theta(\tau))\,d\tau \right. \\ &\left. + (\delta - m - 1) \int_T^{t+T} e^{(\delta-m-1)\int_\tau^T R(\theta(s))\,ds} G(\theta(\tau))\,d\tau \right]^{\frac{1}{\delta-m-1}} \cdot e^{\int_0^t R(\theta(s))\,ds} \\ = &\left[(r_0 e^{\int_0^\tau R(\theta(s))\,ds})^{\delta-m-1} + (\delta - m - 1) \int_0^T e^{(\delta-m-1)\int_\tau^T R(\theta(s))\,ds} G(\theta(\tau))d\tau \right. \\ &\left. + (\delta - m - 1) \int_0^t e^{-(\delta-m-1)\int_0^\tau R(\theta(s))\,ds} G(\theta(\tau))d\tau \right]^{\frac{1}{\delta-m-1}} \cdot e^{\int_0^t R(\theta(s))\,ds}. \end{aligned}$$

By $\int_0^T R(\theta(s))\,ds \cdot \int_0^T e^{(\delta-m-1)\int_\tau^T R(\theta(s))\,ds} G(\theta(\tau))d\tau < 0$ we know that there is a unique $r_0^\star$:

$$r_0^\star = \left[\frac{(\delta - m - 1) \int_0^T e^{(\delta-m-1)\int_\tau^T R(\theta(s))\,ds} G(\theta(\tau))\,d\tau}{1 - e^{(\delta-m-1)\int_0^T R(\theta(s))\,ds}} \right]^{\frac{1}{\delta-m-1}}$$

such that $r(t+T, r_0^\star) = r(t, r_0^\star)$. Therefore, there is only one closed orbit $\theta^\star$ of the vector field $xF(x)+Q(x)$ on the closed surface $S(\theta)$.

We will prove that the closed orbit $\theta^\star$ is an attractor to all trajectories on $S(\theta)$ if $\int_0^T R(\theta(s))\,ds < 0$.

For any initial value $r_0 > r_0^*$, the Poincaré map of the solution $r(t, r_0)$ of equation (2.2a) on the closed surface $S(\theta)$ is

$$\begin{aligned}
&r^{\delta-m-1}(T, r_0) - r_0^{\delta-m-1} \\
&\quad = \left[e^{(\delta-m-1)\int_0^T R(\theta(s))\,ds} - 1\right] r_0^{\delta-m-1} \\
&\qquad + (\delta-m-1)e^{(\delta-m-1)\int_0^T R(\theta(s))\,ds}\int_0^T e^{-(\delta-m-1)\int_0^\tau R(\theta(s))\,ds} G(\theta(\tau))\,d\tau.
\end{aligned}$$

Since $r(T, r_0^\star) = r_0^\star$, we have

$$\begin{aligned}
&(\delta-m-1)e^{(\delta-m-1)\int_0^T R(\theta(s))\,ds}\int_0^T e^{-(\delta-m-1)\int_0^\tau R(\theta(s))\,ds} G(\theta(\tau))\,d\tau \\
&\quad = \left[1 - e^{(\delta-m-1)\int_0^T R(\theta(s))\,ds}\right] r_0^{\star(\delta-m-1)}.
\end{aligned}$$

Then,

$$r^{\delta-m-1}(T, r_0) - r_0^{\delta-m-1} = \left[e^{(\delta-m-1)\int_0^T R(\theta(s))\,ds} - 1\right]\left(r_0^{\delta-m-1} - r_0^{\star(\delta-m-1)}\right) > 0$$

(since $m+1-\delta > 0$, $r_0^{\delta-m-1} - r_0^{\star(\delta-m-1)} < 0$). Therefore, we have $r(T, r_0) < r_0$.

The closed orbit attracts all the trajectories $r(t, r_0)$ on the $S(\theta)$ with initial condition $r(0, r_0) = r_0 > r_0^*$.

For any initial value $r_0 < r_0^*$, similarly, we have

$$r^{\delta-m-1}(T, r_0) - r_0^{\delta-m-1} = \left[e^{(\delta-m-1)\int_0^T R(\theta(s))\,ds} - 1\right]\left(r_0^{\delta-m-1} - r_0^{\star(\delta-m-1)}\right) < 0.$$

Therefore, we have that $r(T, r_0) > r_0$. The closed orbit attracts all the trajectories $r(t, r_0)$ on the $S(\theta)$ with initial condition $r(0, r_0) = r_0 < r_0^*$.

This proves that the global orbit $r(t, r_0^*)\theta(s)$ of the vector field $xF(x)+Q(x)$ is a global attractor of all trajectories of $xF(x)+Q(x)$ on $S(\theta)$ if $\int_0^T R(\theta(s))\,ds < 0$.

Similarly, we can also get that the closed orbit $\theta^\star$ is a repeller for other trajectories on $S(\theta)$ if $\int_0^T R(\theta(s))\,ds > 0$. □

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